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Approximation by Rational Functions: Open Problems*

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In the last half dozen years there have been published some interesting results concerning approximation by rational functions of a complex variable, especially convergence and degree of convergence when the poles of the approximating functions are prescribed indirectly if at all—prescribed, for instance, by extremal properties of the approximating functions. We shall examine a number of such situations, including some results due to D. J. Newman, Turán, Gonçar, Montessus de Ballore, and Walsh. Approximation by polynomials is useful by way of comparison, but the resources of rational functions are much greater, and the corresponding theory much richer.

I. For approximation by polynomials, we have

THEOREM 1. *Let E be a closed bounded point set whose complement K is connected, and regular in the sense that Green's function $G(z)$ for K with pole at infinity exists. Let E_R denote generically the interior of the locus $C_R : G(z) = \log R(>0)$ in K. If $f(z)$ is analytic throughout E_ρ, but throughout no E_R with $R > \rho$, then there exist polynomials $p_n(z)$ in z of respective degrees n such that*

$$\limsup_{n\to\infty}[\max |f(z) - p_n(z)|, z \text{ on } E]^{1/n} = 1/\rho; \tag{1}$$

there exist no such polynomials that the first member of (1) is less than $1/\rho$. The polynomials $p_n(z)$ converge to $f(z)$ uniformly throughout each E_R, $R < \rho$.

A corresponding theorem for approximation by rational functions is much more recent. A rational function of the form

$$r_{n\nu}(z) \equiv \frac{a_0z^n + a_1z^{n-1} + \cdots + a_n}{b_0z^\nu + b_1z^{\nu-1} + \cdots + b_\nu}, \qquad | b_0z^\nu + \cdots + b_\nu | \not\equiv 0,$$

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is said to be *of type* (n, ν). The well-known *Padé table* corresponding to a function $f(z)$, analytic at the origin, consists of the rational functions $r_{n\nu}(z)$ of respective types (n, ν), each of highest order contact with $f(z)$ at $z = 0$:

$$\begin{array}{l} r_{00}(z),\ r_{10}(z),\ r_{20}(z),\ldots \\ r_{01}(z),\ r_{11}(z),\ r_{21}(z),\ldots \\ r_{02}(z),\ r_{12}(z),\ r_{22}(z),\ldots \\ \cdots\cdots\cdots\cdots \end{array}$$

An analog is the table of rational functions $R_{n\nu}(z)$ of respective types (n, ν), each of best approximation to a given function on a given closed bounded set E:

$$\begin{array}{l} R_{00}(z),\ R_{10}(z),\ R_{20}(z),\ldots \\ R_{01}(z),\ R_{11}(z),\ R_{21}(z),\ldots \\ R_{02}(z),\ R_{12}(z),\ R_{22}(z),\ldots \\ \cdots\cdots\cdots\cdots \end{array}$$

An analog of Theorem 1 is, then, [Walsh, 8]:

THEOREM 2. *With the notation of Theorem* 1 *for* E, K, E_R, *and* C_R, *let* $F(z)$ *be analytic on* E, *meromorphic with precisely* ν *poles in* E_ρ, *and let* $R_{n\nu}(z)$ *be the rational functions of types* (n, ν) *of best approximation to* $F(z)$ *on* E. *Then we have for each* ν

$$\limsup_{n\to\infty}[\max |F(z) - R_{n\nu}(z)|,\ z \text{ on } E]^{1/n} = 1/\rho. \tag{2}$$

The functions $R_{n\nu}(z)$ *converge to* $F(z)$ *uniformly throughout each* E_R *with* $R < \rho$, *except in the neighborhoods of the poles of* $F(z)$ *in* E_ρ.

There is obviously an intimate relation between Theorems 1 and 2. Let $F(z)$ be as in Theorem 2, and let the poles of $F(z)$ in E_ρ be $\alpha_1, \alpha_2, \ldots, \alpha_\nu$, with $\pi(z) \equiv (z - \alpha_1) \cdots (z - \alpha_\nu)$. We apply Theorem 1 to the function $f(z) \equiv F(z)\pi(z)$, which satisfies the conditions of Theorem 1; so there exist polynomials in z of respective degrees n satisfying (1). However, $\pi(z)$ is analytic on E, whence

$$\limsup_{n\to\infty}[\max |F(z) - p_n(z)/\pi(z)|,\ z \text{ on } E]^{1/n} \leqslant 1/\rho. \tag{3}$$

The rational functions $p_n(z)/\pi(z)$ are of type (n, ν), hence can be used for comparison as to degree of convergence with the rational functions of best approximation, and (3) implies that the first member of (2) is not greater than $1/\rho$. Use of a generalized form of S. Bernstein's lemma [Walsh, 10] enables one to complete the proof of Theorem 2; there is a one-to-one correspondence between the finite poles of $R_{n\nu}(z)$ and those of $F(z)$ in E_ρ; the former approach the latter. Theorem 2 is, of course, related to the Padé table; each sequence $R_{n\nu}(z)$ is analogous to a row of the Padé table.

Theorem 2 can frequently be applied a number of times with reference to a point set E and a given meromorphic function $f(z)$ by use of various values of ν. For instance, if E is the line segment [2, 3], and $F(z)$ is $\Gamma(z)$, C_ρ may be chosen successively for $\nu = 0, 1, 2,...$ so as to be the ellipse whose foci are $z = 2$ and $z = 3$, passing, respectively, through the poles $z = 0, -1, -2,...$ of $\Gamma(z)$. The corresponding values of ρ are $5 + 2\cdot 6^{1/2}$, $7 + 4\cdot 3^{1/2}$, $9 + 4\cdot 5^{1/2}$, etc. The actual values of $R_{n\nu}(z)$ have been computed numerically by Dr. J. R. Rice, and they verify quite exactly the law (2).

Two other theorems are interesting in comparison with each other, and with Theorems 1 and 2:

THEOREM 3. *Let D be a region bounded by an analytic Jordan curve C, and let the function $f(z)$ be analytic in D, continuous in $D \cup C$. Let $f(z)$ be of class $L(p, \alpha)$ on C, namely such that $f^{(p)}(z)$ exists on C and satisfies there a Lipschitz condition of order α, $0 < \alpha < 1$. Then there exist polynomials $p_n(z)$ of respective degrees n such that on $D \cup C$*

$$|f(z) - p_n(z)| \leqslant A/n^{p+\alpha}, \tag{4}$$

where the constant A does not depend on n or z. Conversely, if $f(z)$ is given and polynomials $p_n(z)$ exist such that (4) is satisfied, with $0 < \alpha < 1$, then $f(z) \in L(p, \alpha)$ on C.

The two parts of Theorem 3 are due to J. H. Curtiss and to Sewell and Walsh, respectively. An analogue of Theorem 3 for rational functions has recently been proved by E. B. Saff [6] in his Ph.D. thesis (1968) at the University of Maryland:

THEOREM 4. *Let D be as in Theorem 3, and let $F(z)$ be now meromorphic with precisely ν poles in D, continuous on C. Then if $F(z)$ is of class $L(p, \alpha)$ on C, there exist rational functions $R_{n\nu}(z)$ of respective types (n, ν) such that $(0 < \alpha < 1)$*

$$|F(z) - R_{n\nu}(z)| \leqslant A/n^{p+\alpha}, \qquad z \text{ on } C. \tag{5}$$

Conversely, if $F(z)$ is given to be continuous on C, meromorphic with precisely ν poles in D, and if rational functions $R_{n\nu}(z)$ of respective types (n, ν) exist, satisfying (5), then $F(z)$ is of class $L(p, \alpha)$ on C.

The first part of Theorem 4 follows from Theorem 3, by precisely the method used in proving the first part of Theorem 2. If the poles of $F(z)$ in D lie in $\alpha_1, \alpha_2, \ldots, \alpha_\nu$, we set

$$\pi(z) \equiv (z - \alpha_1) \cdots (z - \alpha_\nu), \quad f(z) \equiv F(z)\pi(z).$$

Then $f(z)$ is analytic in D, continuous, and of class $L(p, \alpha)$ on C, so by Theorem 3 there exist polynomials $p_n(z)$ such that (4) is satisfied;

$$| F(z)\pi(z) - p_n(z)| \leqslant A/n^{p+\alpha}, \qquad z \text{ on } C.$$

Then (5) follows at once, with $R_{n\nu}(z) \equiv p_n(z)/\pi(z)$ and perhaps with a modification in the constant A.

The second part of Theorem 4, like that of Theorem 2, follows by setting up (for n sufficiently large) a one-to-one correspondence between the ν poles of $R_{n\nu}(z)$ and the ν poles of $F(z)$, for the former approach, respectively, the latter. However, the corresponding method and result are by no means obvious if $F(z)$ and $R_{n\nu}(z)$ have each an infinite number of poles on the point sets involved. *Here lies a largely open problem, namely to extend Theorems* 2 *and* 4 *to the case that* $F(z)$ *has an infinite number of poles in* E_ρ *and in* D, *respectively.*

A somewhat simplified form of this problem is as follows, analogous to Theorem 2. Let $F(z)$ be meromorphic in the entire plane, with a pole α_1 near the point set E. Let a second pole α_2 lie so far from E and α_1 that the pole of the function $R_{11}(z)$ [or $R_{n1}(z)$] of best approximation to $F(z)$ on E is "near" the point α_1, and let a third pole α_3 lie so far from E, α_1, and α_2 that the two poles of the function $R_{22}(z)$ [or $R_{n2}(z)$] of best approximation to $F(z)$ on E are "near" the points α_1 and α_2, and so on. Then it is to be conjectured that

$$\limsup_{n\to\infty}[\max | F(z) - R_{nn}(z)|, z \text{ on } E]^{1/n} = 0,$$

and that the $R_{nn}(z)$ converge to $F(z)$ on each compact set containing no poles of $F(z)$. However, even this mild conjecture has not yet been accurately formulated and proved.

II. We turn now to another kind of result, contrasting approximation by rational functions with approximation by polynomials. Dunham Jackson showed in work now classical that the best polynomial approximation to the function $| x |$ on $[-1, 1]$ is of the order $1/n$, in the sense that for each n there exists a polynomial $P_n(x)$ of degree n such that

$$|\, | x | - P_n(x)| \leqslant C_1/n, \qquad x \text{ on } [-1, 1], \tag{6}$$

but (S. Bernstein) there exists no sequence of polynomials $P_n(x)$ of respective degrees n such that for some α, $1 < \alpha$, we have

$$| \, | x | - P_n(x)| \leqslant C_2/n^{\alpha}, \qquad x \text{ on } [-1, 1].$$

However, for approximation by rational functions $R_n(x)$ of respective degrees n, D. J. Newman [5] has shown the surprising result that there exist $R_n(x)$ with $(n > 4)$

$$|| x | - R_n(x)| \leqslant 3e^{-\sqrt{n}}, \qquad x \text{ on } [-1, 1], \tag{7}$$

but there exist no such rational functions such that

$$|| x | - R_n(x)| \leqslant e^{-9\sqrt{n}}/2, \qquad x \text{ on } [-1, 1]. \tag{8}$$

His method is based on explicit formulas involving the exponential function, and it has been extended and sharpened to include approximation of various other given functions by rational functions.

After Newman, Szüsz and Turán [7] investigated approximation to functions with convex higher derivatives satisfying a Lipschitz condition, and also to piecewise analytic functions. They also established the order $|f(x) - r_n(x)| \leqslant e^{-c\sqrt{n}}$ for rational approximation. G. Freud [2] found for rational approximation to Lipschitz functions of order α $(0 < \alpha < 1)$ a degree of approximation $O(n^{-1} \log^2 n)$, better than that for polynomials. A. A. Gonçar [3] used methods of conformal mapping, and other methods going back to Zolotareff (1877) to study approximation to piecewise analytic functions, and also [4] used the modulus of continuity $\omega(\delta)$ on [0, 1] and analyticity in the region $| z - 1 | < 1$ to obtain rational functions approaching the given $f(x)$ with an order $\omega(\exp(-cn^{1/2}))$, on the interval [0, 1], an improvement over Jackson.

The results already obtained are highly surprising, but require considerable further investigation before a complete analog of the Jackson–Bernstein theory is developed, both on an interval of the real axis and on an arbitrary Jordan arc. The essence of the problem is: Where should the poles of the approximating rational functions be placed for maximum degree of convergence?

As Newman states, there is a widespread folk theorem to the effect that: "In some overall sense, rational approximation is essentially no better than polynomial approximation". I have sharpened [12] this naive rough observation by proving through use of approximation by bounded analytic functions:

If the function $f(z)$ is approximable on a Jordan arc C of the z-plane, to the order $n^{-\alpha}(\alpha > 0)$ by rational functions $Q_n(z)$ of respective degrees n whose poles have no limit point on C, then $f(z)$ is also approximable on C to the order $n^{-\alpha}$ by polynomials $p_n(z)$ of respective degrees n.

In Newman's case, this theorem applies to every subinterval of $[-1, 1]$ which contains the origin, so the origin is a limit point of poles of Newman's rational functions, and indeed of poles of any set of rational functions of respective degrees n that improves (6) to n^α.

III. In a physical problem, under suitable circumstances an unknown function may frequently be related to a continued fraction, a differential equation, a power series, or some other algorithm leading to a sequence of rational functions. Again the question arises, where are the poles of the approximating rational functions, say the Padé approximants of a continued fraction? As an example, Theorem 2 is analogous to an older theorem on the Padé table due to Montessus de Ballore. A fairly large body of literature has now grown up around this general topic, largely due to physicists, and with a large number of conclusions based on admittedly unusual hypotheses. As an illustration, we mention one of the newer theorems [Walsh, 11]:

THEOREM 5. *Let the function $f(z)$ be analytic at the origin and on $\Gamma : |z| = \rho(> 0)$, meromorphic with precisely ν poles in $\Delta : |z| < \rho$, and suppose the Padé approximants $P_{nn}(z)$ are bounded on Γ:*

$$|f(z) - P_{nn}(z)| \leqslant M, \qquad z \text{ on } \Gamma. \tag{9}$$

Suppose $P_{nn}(z)$ has precisely N_n poles in Δ, with $N_n/n \to 0$. Then we have

$$\limsup_{n\to\infty}[\max |f(z) - P_{nn}(z)|, z \text{ on } T]^{1/n} \leqslant [\max |z|, z \text{ on } T]^2/\rho^2, \tag{10}$$

where T is an arbitrary closed set in Δ containing no limit points of poles of $P_{nn}(z)$.

The weakness of this theorem is that it may be difficult to verify that Δ contains precisely N_n poles of $P_{nn}(z)$ with $N_n/n \to 0$, and that T contains no limit point of poles of $P_{nn}(z)$.

Baker [1] has recently published a report on this area of research. He himself has a sequence of theorems and corollaries which he calls quasitheorems and quasicorollaries. These are based on a conjecture which is unproved but to which there are no known counterexamples:

Conjecture (*Baker*). If $P(z)$ is a power series representing a function $f(z)$ which is analytic for $|z| \leqslant 1$ except for m poles within this circle and except for $z = +1$, at which point $f(z)$ is assumed continuous when only points $|z| \leqslant 1$ are considered, then at least a subsequence of the $[N, N]$ Padé approximants converges uniformly to $f(z)$ in the region formed from $|z| < 1$ by removing small open disks with centers at these poles.

Based on this conjecture, Baker proves various quasitheorems and quasicorollaries. By "convergence" of a sequence he means that at least a subsequence converges. Proof or disproof of the Conjecture is a prominent open question.

In this complex of problems relating to Padé approximants, an interesting result [Walsh, 9] is that *if $f(z)$ is analytic in the neighborhood of the origin, the Padé approximant of type (n, ν) is the limit of the rational function of type (n, ν) of best approximation to $f(z)$ on the point set $|z| \leqslant \epsilon$ as ϵ approaches zero.* Thus the relation between the theory of Padé approximation and that of best approximation is far closer than that of mere analogy. The question for both theories still exists: Where are the poles of the approximating functions?

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